# Highly Undecidable Problems For Infinite Computations

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# HIGHLY UNDECIDABLE PROBLEMS FOR INFINITE COMPUTATIONS

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**Abstract**. We show that many classical decision problems about 1-counter $\omega$-languages, context free $\omega$-languages, or infinitary rational relations, are $\Pi_2^1$-complete, hence located at the second level of the analytical hierarchy, and "highly undecidable". In particular, the universality problem, the inclusion problem, the equivalence problem, the determinizability problem, the complementability problem, and the unambiguity problem are all $\Pi_2^1$-complete for context-free $\omega$-languages or for infinitary rational relations. Topological and arithmetical properties of 1-counter $\omega$-languages, context free $\omega$-languages, or infinitary rational relations, are also highly undecidable. These very surprising results provide the first examples of highly undecidable problems about the behaviour of very simple finite machines like 1-counter automata or 2-tape automata.

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## 1. Introduction

Many classical decision problems arise naturally in the fields of Formal Language Theory and of Automata Theory. When languages of finite words are considered it is well known that most problems about regular languages accepted by finite automata are decidable. On the other hand, at the second level of the Chomsky Hierarchy, most problems about context-free languages accepted by pushdown automata or generated by context-free grammars are undecidable. For instance it follows from the undecidability of the Post Correspondence Problem that the

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universality problem, the inclusion and the equivalence problems for context-free languages are also undecidable. Notice that some few problems about context-free languages remain decidable like the following ones: "Is a given context-free language $L$ empty ? " "Is a given context-free language $L$ infinite ? " "Does a given word $x$ belong to a given context-free language $L$ ? " Sénizergues proved in [Sén01] that the difficult problem of the equivalence of two deterministic pushdown automata is decidable. Another problem about finite simple machines is the equivalence problem for deterministic multitape automata. It has been proved to be decidable by Harju and Karhumäki in [HK91]. But all known problems about acceptance by Turing machines are undecidable, [HMU01].
Languages of infinite words accepted by finite automata were first studied by Büchi to prove the decidability of the monadic second order theory of one successor over the integers. Since then regular $\omega$-languages have been much studied and many applications have been found for specification and verification of non-terminating systems, see [Tho90, Sta97, PP04] for many results and references. More powerful machines, like pushdown automata, Turing machines, have also been considered for the reading of infinite words, see Staiger's survey [Sta97] and the fundamental study [EH93] of Engelfriet and Hoogeboom on $\mathbf{X}$-automata, i.e. finite automata equipped with a storage type $\mathbf{X}$. As in the case of finite words, most problems about regular $\omega$-languages have been shown to be decidable. On the other hand most problems about context-free $\omega$-languages are known to be undecidable, [CG77]. Notice that almost all undecidability proofs rely on the undecidability of the Post Correspondence Problem which is complete for the class of recursively enumerable problems, i.e. complete at the first level of the arithmetical hierarchy. Thus undecidability results about context-free $\omega$-languages provided only hardness results for the first level of the arithmetical hierarchy.
Castro and Cucker studied decision problems for $\omega$-languages of Turing machines in [CC89]. They studied the degrees of many classical decision problems like : "Is the $\omega$-language recognized by a given machine non empty ?", "Is it finite ?" "Do two given machines recognize the same $\omega$-language ?"
Their motivation was on one side to classify the problems about Turing machines and on the other side to "give natural complete problems for the lowest levels of the analytical hierarchy which constitute an analog of the classical complete problems given in recursion theory for the arithmetical hierarchy".
On the other hand we showed in [Fin06a] that context free $\omega$-languages, or even $\omega$-languages accepted by Büchi 1-counter automata, have the same topological complexity as $\omega$-languages accepted by Turing machines with a Büchi acceptance condition. We use in this paper several constructions of [Fin06a] to infer some undecidability results from those of [CC89]. Notice that one cannot infer directly from topological results of [Fin06a] that the degrees of decision problems for $\omega$-languages of Büchi 1-counter automata are the same as the degrees of the corresponding decision problems about Turing machines. For instance the non-emptiness problem and the infiniteness problem are decidable for $\omega$-languages accepted by Büchi 1-counter automata or even by Büchi pushdown automata but the non-emptiness problem and the infiniteness problem for $\omega$-languages of Turing machines are both

$\Sigma_1^1$-complete, hence highly undecidable, [CC89]. However we can show that many other classical decision problems about 1-counter $\omega$-languages or context free $\omega$-languages, are $\Pi_2^1$-complete, hence located at the second level of the analytical hierarchy, and "highly undecidable". In particular, the universality problem, the inclusion problem, the equivalence problem, the determinizability problem, the complementability problem, and the unambiguity problem are all $\Pi_2^1$-complete for $\omega$-languages of Büchi 1-counter automata. Topological and arithmetical properties of 1-counter $\omega$-languages and of context free $\omega$-languages are also highly undecidable.
In another paper we had also shown that infinitary rational relations accepted by 2-tape Büchi automata have the same topological complexity as $\omega$-languages accepted by Büchi 1-counter automata or by Büchi Turing machines. This very surprising result was obtained by using a simulation of the behaviour of real time 1-counter automata by 2-tape Büchi automata, [Fin06b]. Using some constructions of [Fin06b] we infer from results about degrees of decision problems for Büchi 1-counter automata some very similar results about decision problems for infinitary rational relations accepted by 2-tape Büchi automata.
These very surprising results provide the first examples of highly undecidable problems about the behaviour of very simple finite machines like 1-counter automata or 2-tape automata.

The paper is organized as follows. In Section 2 we recall some notions about arithmetical and analytical hierarchies and also about the Borel hierarchy. We study decision problems for infinite computations of 1-counter automata in Section 3. We infer some corresponding results about infinite computations of 2-tape automata in Section 4. Some concluding remarks are given in Section 5.

## 2. Arithmetical and analytical hierarchies

### 2.1. Hierarchies of sets of integers

The set of natural numbers is denoted by $\mathbb{N}$ and the set of all total functions from $\mathbb{N}$ into $\mathbb{N}$ will be denoted by $\mathcal{F}$.

We assume the reader to be familiar with the arithmetical hierarchy on subsets of $\mathbb{N}$. We now recall the notions of analytical hierarchy and of complete sets for classes of this hierarchy which may be found in [Rog67]; see also for instance [Odi89,Odi99] for more recent textbooks on computability theory.

**Definition 2.1.** *Let $k, l > 0$ be some integers. $\Phi$ is a partial computable functional of $k$ function variables and $l$ number variables if there exists $z \in \mathbb{N}$ such that for any $(f_1, \ldots, f_k, x_1, \ldots, x_l) \in \mathcal{F}^k \times \mathbb{N}^l$, we have*

$$\Phi(f_1, \ldots, f_k, x_1, \ldots, x_l) = \tau_z^{f_1,\ldots,f_k}(x_1, \ldots, x_l),$$

*where the right hand side is the output of the Turing machine with index $z$ and oracles $f_1, \dots, f_k$ over the input $(x_1, \dots, x_l)$. For $k > 0$ and $l = 0$, $\Phi$ is a partial computable functional if, for some $z$,*

$$\Phi(f_1, \dots, f_k) = \tau_z^{f_1,\dots,f_k}(0).$$

*The value $z$ is called the Gödel number or index for $\Phi$.*

**Definition 2.2.** *Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$. The relation $R$ is said to be a computable relation of $k$ function variables and $l$ number variables if its characteristic function is computable.*

We now define analytical subsets of $\mathbb{N}^l$.

**Definition 2.3.** *A subset $R$ of $\mathbb{N}^l$ is analytical if it is computable or if there exists a computable set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that*

$$R = \{(x_1, \dots, x_l) \mid (Q_1 s_1)(Q_2 s_2) \dots (Q_{m+n-l} s_{m+n-l}) S(f_1, \dots, f_m, x_1, \dots, x_n)\},$$

*where $Q_i$ is either $\forall$ or $\exists$ for $1 \leq i \leq m+n-l$, and where $s_1, \dots, s_{m+n-l}$ are $f_1, \dots, f_m, x_{l+1}, \dots, x_n$ in some order.*
*The expression $(Q_1 s_1)(Q_2 s_2) \dots (Q_{m+n-l} s_{m+n-l}) S(f_1, \dots, f_m, x_1, \dots, x_n)$ is called a predicate form for $R$. A quantifier applying over a function variable is of type $1$, otherwise it is of type $0$. In a predicate form the (possibly empty) sequence of quantifiers, indexed by their type, is called the prefix of the form. The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type $0$ from the prefix.*

We can now distinguish the levels of the analytical hierarchy by considering the number of alternations in the reduced prefix.

**Definition 2.4.** *For $n > 0$, a $\Sigma_n^1$-prefix is one whose reduced prefix begins with $\exists^1$ and has $n-1$ alternations of quantifiers. A $\Sigma_0^1$-prefix is one whose reduced prefix is empty. For $n > 0$, a $\Pi_n^1$-prefix is one whose reduced prefix begins with $\forall^1$ and has $n-1$ alternations of quantifiers. A $\Pi_0^1$-prefix is one whose reduced prefix is empty.*
*A predicate form is a $\Sigma_n^1$ ($\Pi_n^1$)-form if it has a $\Sigma_n^1$ ($\Pi_n^1$)-prefix. The class of sets in some $\mathbb{N}^l$ which can be expressed in $\Sigma_n^1$-form (respectively, $\Pi_n^1$-form) is denoted by $\Sigma_n^1$ (respectively, $\Pi_n^1$).*
*The class $\Sigma_0^1 = \Pi_0^1$ is the class of arithmetical sets.*

We now recall some well known results about the analytical hierarchy.

**Proposition 2.5.** *Let $R \subseteq \mathbb{N}^l$ for some integer $l$. Then $R$ is an analytical set iff there is some integer $n \geq 0$ such that $R \in \Sigma_n^1$ or $R \in \Pi_n^1$.*

**Theorem 2.6.** *For each integer $n \geq 1$,*

(a) $\Sigma_n^1 \cup \Pi_n^1 \subsetneq \Sigma_{n+1}^1 \cap \Pi_{n+1}^1$.
(b) *A set $R \subseteq \mathbb{N}^l$ is in the class $\Sigma_n^1$ iff its complement is in the class $\Pi_n^1$.*

(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$.

Transformations of prefixes are often used, following the rules given by the next theorem.

**Theorem 2.7.** *For any predicate form with the given prefix, an equivalent predicate form with the new one can be obtained, following the allowed prefix transformations given below :*

(a) $\ldots \exists^0 \exists^0 \ldots \rightarrow \ldots \exists^0 \ldots,$
$\ldots \forall^0 \forall^0 \ldots \rightarrow \ldots \forall^0 \ldots;$
(b) $\ldots \exists^1 \exists^1 \ldots \rightarrow \ldots \exists^1 \ldots,$
$\ldots \forall^1 \forall^1 \ldots \rightarrow \ldots \forall^1 \ldots;$
(c) $\ldots \exists^0 \ \ldots \rightarrow \ldots \exists^1 \ldots,$
$\ldots \forall^0 \ \ldots \rightarrow \ldots \forall^1 \ldots;$
(d) $\ldots \exists^0 \forall^1 \ldots \rightarrow \ldots \forall^1 \exists^0 \ldots,$
$\ldots \forall^0 \exists^1 \ldots \rightarrow \ldots \exists^1 \forall^0 \ldots;$

We can now define the notion of 1-reduction and of $\Sigma^1_n$-complete (respectively, $\Pi^1_n$-complete) sets. Notice that we give the definition for subsets of $\mathbb{N}$ but this can be easily extended to subsets of $\mathbb{N}^l$ for some integer $l$.

**Definition 2.8.** *Given two sets $A, B \subseteq \mathbb{N}$ we say $A$ is 1-reducible to $B$ and write $A \leq_1 B$ if there exists a total computable injective function $f$ from $\mathbb{N}$ to $\mathbb{N}$ with $A = f^{-1}[B]$.*

**Definition 2.9.** *A set $A \subseteq \mathbb{N}$ is said to be $\Sigma^1_n$-complete (respectively, $\Pi^1_n$-complete) iff $A$ is a $\Sigma^1_n$-set (respectively, $\Pi^1_n$-set) and for each $\Sigma^1_n$-set (respectively, $\Pi^1_n$-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$.*

For each integer $n \geq 1$ there exist some $\Sigma^1_n$-complete subset of $\mathbb{N}$. Such sets are precisely defined in [Rog67] or [CC89].

**Notation 2.10.** *$U_n$ denotes a $\Sigma^1_n$-complete subset of $\mathbb{N}$. The set $U_n^- = \mathbb{N} - U_n \subseteq \mathbb{N}$ is a $\Pi^1_n$-complete set.*

## 2.2. Hierarchies of sets of infinite words

We assume now the reader to be familiar with the theory of formal ($\omega$)-languages [Tho90, Sta97]. We shall follow usual notations of formal language theory.
When $\Sigma$ is a finite alphabet, a *non-empty finite word* over $\Sigma$ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$ , and $k$ is an integer $\geq 1$. The *length* of $x$ is $k$, denoted by $|x|$. The *empty word* has no letter and is denoted by $\lambda$; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over $\Sigma$.
The *first infinite ordinal* is $\omega$. An $\omega$-*word* over $\Sigma$ is an $\omega$ -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When $\sigma$ is an $\omega$-word over $\Sigma$, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$, where for all $i$, $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.
The usual concatenation product of two finite words $u$ and $v$ is denoted $u.v$ (and sometimes just $uv$). This product is extended to the product of a finite word $u$

and an $\omega$-word $v$: the infinite word $u.v$ is then the $\omega$-word such that:
$(u.v)(k) = u(k)$ if $k \leq |u|$ , and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.
The *set of* $\omega$-*words* over the alphabet $\Sigma$ is denoted by $\Sigma^\omega$. An $\omega$-*language* over an alphabet $\Sigma$ is a subset of $\Sigma^\omega$. The complement (in $\Sigma^\omega$) of an $\omega$-language $V \subseteq \Sigma^\omega$ is $\Sigma^\omega - V$, denoted $V^-$.

We assume now the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Kec95, Sta97, PP04]. There is a natural metric on the set $\Sigma^\omega$ of infinite words over a finite alphabet $\Sigma$ containing at least two letters which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer $n$ such that the $(n+1)^{st}$ letter of $u$ is different from the $(n+1)^{st}$ letter of $v$. This metric induces on $\Sigma^\omega$ the usual Cantor topology for which *open subsets* of $\Sigma^\omega$ are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set. Define now the *Borel Hierarchy* of subsets of $\Sigma^\omega$:

**Definition 2.11.** *For a non-null countable ordinal $\alpha$, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space $\Sigma^\omega$ are defined as follows:*
*$\mathbf{\Sigma}^0_1$ is the class of open subsets of $\Sigma^\omega$,*
*$\mathbf{\Pi}^0_1$ is the class of closed subsets of $\Sigma^\omega$,*
*and for any countable ordinal $\alpha \geq 2$:*
*$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of $\Sigma^\omega$ in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.*
*$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of $\Sigma^\omega$ in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.*

For a countable ordinal $\alpha$, a subset of $\Sigma^\omega$ is a Borel set of *rank* $\alpha$ iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There are also some subsets of $\Sigma^\omega$ which are not Borel. In particular the class of Borel subsets of $\Sigma^\omega$ is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets.
We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with $Y$ a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_1$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$. $\mathbf{\Sigma}^0_n$ (respectively $\mathbf{\Pi}^0_n$)-complete sets, with $n$ an integer $\geq 1$, are thoroughly characterized in [Sta86].

We recall now the definition of the arithmetical hierarchy of $\omega$-languages which form the effective analogue to the hierarchy of Borel sets of finite ranks.
Let $X$ be a finite alphabet. An $\omega$-language $L \subseteq X^\omega$ belongs to the class $\Sigma_n$ if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where $Q_i$ is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An $\omega$-language $L \subseteq X^\omega$ belongs to the class $\Pi_n$ if and only if its complement

$X^\omega - L$ belongs to the class $\Sigma_n$. The inclusion relations that hold between the classes $\Sigma_n$ and $\Pi_n$ are the same as for the corresponding classes of the Borel hierarchy. The classes $\Sigma_n$ and $\Pi_n$ are included in the respective classes $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ of the Borel hierarchy, and cardinality arguments suffice to show that these inclusions are strict.

As in the case of the Borel hierarchy, projections of arithmetical sets lead beyond the arithmetical hierarchy, to the analytical hierarchy of $\omega$-languages. The first class of this hierarchy is the (lightface) class $\Sigma^1_1$ of *effective analytic sets* which are obtained by projection of arithmetical sets.
In fact an $\omega$-language $L \subseteq X^\omega$ is in the class $\Sigma^1_1$ iff it is the projection of an $\omega$-language over the alphabet $X \times \{0,1\}$ which is in the class $\Pi_2$. The (lightface) class $\Pi^1_1$ of *effective co-analytic sets* is simply the class of complements of effective analytic sets. We denote as usual $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.

The Borel ranks of (lightface) $\Delta^1_1$ sets are the (recursive) ordinals $\gamma < \omega_1^{\mathrm{CK}}$, where $\omega_1^{\mathrm{CK}}$ is the first non-recursive ordinal, usually called the Church-Kleene ordinal. Moreover, for every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete sets in the class $\Delta^1_1$.

## 3. Infinite computations of 1-counter automata

Recall the notion of acceptance of infinite words by Turing machines considered by Castro and Cucker in [CC89].

**Definition 3.1.** *A non deterministic Turing machine $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$, where $Q$ is a finite set of states, $\Sigma$ is a finite input alphabet, $\Gamma$ is a finite tape alphabet satisfying $\Sigma \subseteq \Gamma$, $q_0$ is the initial state, and $\delta$ is a mapping from $Q \times \Gamma$ to subsets of $Q \times \Gamma \times \{L, R, S\}$. A configuration of $\mathcal{M}$ is a triple $(q, \sigma, i)$, where $q \in Q$, $\sigma \in \Gamma^\omega$ and $i \in \mathbb{N}$. An infinite sequence of configurations $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on $w \in \Sigma^\omega$ iff:*
(a) $(q_1, \alpha_1, j_1) = (q_0, w, 1)$, *and*
(b) *for each* $i \geq 1$, $(q_i, \alpha_i, j_i) \vdash (q_{i+1}, \alpha_{i+1}, j_{i+1})$,

*where $\vdash$ is the transition relation of $\mathcal{M}$ defined as usual. The run $r$ is said to be complete if $(\forall n \geq 1)(\exists k \geq 1)(j_k \geq n)$. The run $r$ is said to be oscillating if $(\exists k \geq 1)(\forall n \geq 1)(\exists m \geq n)(j_m = k)$.*

**Definition 3.2.** *Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non deterministic Turing machine and $F \subseteq Q$. The $\omega$-language accepted by $(\mathcal{M}, F)$ is the set of $\omega$-words $\sigma \in \Sigma^\omega$ such that there exists a complete non oscillating run $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ of $\mathcal{M}$ on $\sigma$ such that, for all $i, q_i \in F$.*

The above acceptance condition is denoted $1'$-acceptance in [CG78b]. Another usual acceptance condition is the now called Büchi acceptance condition which is also denoted 2-acceptance in [CG78b]. We just now recall its definition.

**Definition 3.3.** *Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non deterministic Turing machine and $F \subseteq Q$. The $\omega$-language Büchi accepted by $(\mathcal{M}, F)$ is the set of $\omega$-words $\sigma \in \Sigma^\omega$ such that there exists a complete non oscillating run $r = (q_i, \alpha_i, j_i)_{i\geq 1}$ of $\mathcal{M}$ on $\sigma$ and infinitely many integers $i$ such that $q_i \in F$.*

Recall that Cohen and Gold proved in [CG78b, Theorem 8.6] that one can effectively construct, from a given non deterministic Turing machine, another equivalent (i.e., accepting the same $\omega$-language) non deterministic Turing machine, equipped with the same kind of acceptance condition, and in which every run is complete non oscillating.

Cohen and Gold proved also in [CG78b, Theorem 8.2] that an $\omega$-language is accepted by a non deterministic Turing machine with $1'$-acceptance condition iff it is accepted by a non deterministic Turing machine with Büchi acceptance condition. It is known that $\omega$-languages accepted by non deterministic Turing machines with $1'$ or Büchi acceptance condition form the (lightface) class $\Sigma^1_1$ of effective analytic sets, [Sta97].

We now recall the definition of $k$-counter Büchi automata which will be useful in the sequel.

**Definition 3.4.** *Let $k$ be an integer $\geq 1$. A $k$-counter machine ($k$-CM) is a 4-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0)$, where $K$ is a finite set of states, $\Sigma$ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0, 1\}^k \times K \times \{0, 1, -1\}^k$ is the transition relation. The $k$-counter machine $\mathcal{M}$ is said to be real time iff: $\Delta \subseteq K \times \Sigma \times \{0, 1\}^k \times K \times \{0, 1, -1\}^k$, i.e. iff there is no $\lambda$-transitions.*
*If the machine $\mathcal{M}$ is in state $q$ and $c_i \in \mathbf{N}$ is the content of the $i^{th}$ counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k + 1)$-tuple $(q, c_1, \ldots, c_k)$.*

*For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:*

$$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$$

*Thus we see that the transition relation must satisfy:*
*if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$, then $j_m = 0$ or $j_m = 1$ (but $j_m$ may not be equal to $-1$).*

*Let $\sigma = a_1a_2 \ldots a_n \ldots$ be an $\omega$-word over $\Sigma$. An $\omega$-sequence of configurations $r = (q_i, c^i_1, \ldots c^i_k)_{i\geq 1}$ is called a run of $\mathcal{M}$ on $\sigma$, starting in configuration $(p, c_1, \ldots, c_k)$, iff:*

1. $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$
2. *for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$ and such that either $a_1a_2 \ldots a_n \ldots = b_1b_2 \ldots b_n \ldots$ or $b_1b_2 \ldots b_n \ldots$ is a finite prefix of $a_1a_2 \ldots a_n \ldots$*

*The run $r$ is said to be complete when $a_1a_2 \ldots a_n \ldots = b_1b_2 \ldots b_n \ldots$*
*For every such run,* $\mathrm{In}(r)$ *is the set of all states entered infinitely often during run $r$.*
*A complete run $r$ of $M$ on $\sigma$, starting in configuration $(q_0, 0, \ldots, 0)$, will be simply called "a run of $M$ on $\sigma$".*

**Definition 3.5.** *A Büchi $k$-counter automaton is a 5-tuple $\mathcal{M}=(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'=(K, \Sigma, \Delta, q_0)$ is a $k$-counter machine and $F \subseteq K$ is the set of accepting states. The $\omega$-language accepted by $\mathcal{M}$ is*

$L(\mathcal{M})= \{\sigma \in \Sigma^\omega \mid$ *there exists a run $r$ of $\mathcal{M}$ on $\sigma$ such that* $\mathrm{In}(r) \cap F \neq \emptyset\}$

The class of $\omega$-languages accepted by Büchi $k$-counter automata will be denoted $\mathbf{BCL}(k)_\omega$. The class of $\omega$-languages accepted by *real time* Büchi $k$-counter automata will be denoted $\mathbf{r}$-$\mathbf{BCL}(k)_\omega$.

Remark that 1-counter automata introduced above are equivalent to pushdown automata whose stack alphabet is in the form $\{Z_0, A\}$ where $Z_0$ is the bottom symbol which always remains at the bottom of the stack and appears only there and $A$ is another stack symbol.
The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free $\omega$-languages accepted by Büchi pushdown automata.

Using a standard construction exposed for instance in [HMU01] we can construct, from a Büchi Turing machine, an equivalent 2-counter automaton accepting the same $\omega$-language with a Büchi acceptance condition.

Notice that these constructions are effective and that they can be achieved in an injective way. So we can now state the following lemma.

**Lemma 3.6.** *There is an injective computable function $H_1$ from $\mathbb{N}$ into $\mathbb{N}$ satisfying the following property.*
*If $\mathcal{M}_z$ is the non deterministic Turing machine (equipped with a $1'$-acceptance condition) of index $z$, and if $\mathcal{A}_{H_1(z)}$ is the 2-counter automaton (equipped with a 2-acceptance condition) of index $H_1(z)$, then these two machines accept the same $\omega$-language, i.e. $L(\mathcal{M}_z) = L(\mathcal{A}_{H_1(z)})$.*

We are now going to recall some constructions which were used in [Fin06a] in the study of topological properties of context-free $\omega$-languages.

Let $\Sigma$ be an alphabet having at least two letters, $E$ be a new letter not in $\Sigma$, $S$ be an integer $\geq 1$, and $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ be the function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4) \ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}} \ldots$$

It is proved in [Fin06a] that if $L \subseteq \Sigma^\omega$ is an $\omega$-language in the class $\mathbf{BCL}(2)_\omega$ and $k = cardinal(\Sigma) + 2$, $S = (3k)^3$, then one can construct effectively, from a Büchi 2-counter automaton $\mathcal{B}$ accepting $L$, a real time Büchi 8-counter automaton $\mathcal{A}$ such that $L(\mathcal{A}) = \theta_S(L)$, so $\theta_S(L)$ is in the class **r-BCL**$(8)_\omega$. This construction can be made injective. On the other hand, it is easy to see that $\theta_S(\Sigma^\omega)^- = (\Sigma \cup \{E\})^\omega - \theta_S(\Sigma^\omega)$ is accepted by a real time Büchi 1-counter automaton. The class **r-BCL**$(8)_\omega$ is closed by finite union in an effective way, so $\theta_S(L) \cup \theta_S(\Sigma^\omega)^-$ is accepted by a real time Büchi 8-counter automaton which can be effectively constructed from $\mathcal{B}$. Thus we get the following result:

**Lemma 3.7.** *There is an injective computable function $H_2$ from $\mathbb{N}$ into $\mathbb{N}$ satisfying the following property.*
*If $\mathcal{B}_z$ is the Büchi 2-counter automaton (reading words over $\Sigma$) of index $z$, and if $\mathcal{A}_{H_2(z)}$ is the real time Büchi 8-counter automaton of index $H_2(z)$, then $L(\mathcal{A}_{H_2(z)}) = \theta_S(L(\mathcal{B}_z)) \cup \theta_S(\Sigma^\omega)^-$.*

Another coding has been used in [Fin06a] which we now recall. Let $K = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9699690$ be the product of the eight first prime numbers. Then an $\omega$-word $x \in \Gamma^\omega$ is coded by the $\omega$-word

$$h_K(x) = A.0^K.x(1).B.0^{K^2}.A.0^{K^2}.x(2).B.0^{K^3}.A.0^{K^3}.x(3).B \ldots B.0^{K^n}.A.0^{K^n}.x(n).B \ldots$$

over the alphabet $\Gamma \cup \{A, B, 0\}$, where $A, B, 0$ are new letters not in $\Gamma$. It is proved in [Fin06a] that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting $L(\mathcal{A}) \subseteq \Gamma^\omega$, one can effectively construct (in an injective manner) a Büchi 1-counter automaton accepting the $\omega$-language $h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-$.

Consider now the mapping $\phi_K : (\Gamma \cup \{A, B, 0\})^\omega \to (\Gamma \cup \{A, B, F, 0\})^\omega$ which is simply defined by: for all $x \in (\Gamma \cup \{A, B, 0\})^\omega$,

$$\phi_K(x) = F^{K-1}.x(1).F^{K-1}.x(2) \ldots F^{K-1}.x(n).F^{K-1}.x(n+1).F^{K-1} \ldots$$

Then the $\omega$-language $\phi_K(h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-)$ is accepted by a real time Büchi 1-counter automaton which can be effectively constructed from the real time Büchi 8-counter automaton $\mathcal{A}$. On the other hand it is easy to see that the $\omega$-language $(\Gamma \cup \{A, B, F, 0\})^\omega - \phi_K((\Gamma \cup \{A, B, 0\})^\omega)$ is $\omega$-regular and to construct a Büchi automaton accepting it. Then one can effectively construct from $\mathcal{A}$ a real time Büchi 1-counter automaton accepting the $\omega$-language $\phi_K(h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, 0\})^\omega)^-$. This can be done in an injective manner. So we can state the following lemma.

**Lemma 3.8.** *There is an injective computable function $H_3$ from $\mathbb{N}$ into $\mathbb{N}$ satisfying the following property.*
*If $A_z$ is the real time Büchi 8-counter automaton (reading words over $\Gamma$) of index $z$, and if $A_{H_3(z)}$ is the real time Büchi 1-counter automaton of index $H_3(z)$ (reading words over $\Gamma \cup \{A, B, F, 0\}$), then :*

$$L(A_{H_3(z)}) = \phi_K(h_K(L(\mathcal{A}_z)) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, 0\})^\omega)^-$$

In the sequel we shall consider, as in [CC89], that $\Sigma$ contains only two letters and we denote these letters by $a$ and $b$ so $\Sigma = \{a, b\}$. Then $\Gamma = \Sigma \cup \{E\}$ and we set $\Omega = \Gamma \cup \{A, B, F, 0\} = \{a, b, E, A, B, F, 0\}$.

From now on, we shall denote $\mathcal{M}_z$ the non deterministic Turing machine of index $z$, (reading words over $\Sigma$), equipped with a $1'$-acceptance condition, and $\mathcal{C}_z$ the real time Büchi 1-counter automaton of index $z$ (reading words over $\Omega$).

We set $H = H_3 \circ H_2 \circ H_1$, where $H_1$, $H_2$, and $H_3$ are the computable functions from $\mathbb{N}$ into $\mathbb{N}$ described above, the functions $H_1$, $H_2$ and $H_3$ being given by Lemmas 3.6, 3.7, and 3.8. Thus $H$ is an injective computable function from $\mathbb{N}$ into $\mathbb{N}$ and if $z$ is the index of a non deterministic Turing machine reading words over $\Sigma$ and equipped with a $1'$-acceptance condition, then $H(z)$ is the index of a non deterministic real time Büchi 1-counter automaton reading words over the alphabet $\Omega = \{a, b, E, A, B, F, 0\}$.

Notice also that a run $r$ of a real time Büchi 1-counter automaton may be easily coded by an infinite word over the alphabet $\{0, 1\}$. We can then identified $r$ with its code $\bar{r} \in \{0, 1\}^\omega$. Then it is easy to see that "$r$ is a run of $\mathcal{C}_z$ over the $\omega$-word $\sigma \in \Omega^\omega$" and "$r$ is an accepting run" can be expressed by arithmetical formulas.

We can now state that the universality problem for $\omega$-languages of real time Büchi 1-counter automata is highly undecidable.

**Theorem 3.9.** *The universality problem for $\omega$-languages of real time Büchi 1-counter automata is $\Pi^1_2$-complete, i.e. the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z) = \Omega^\omega\}$ is $\Pi^1_2$-complete.*

**Proof.** We prove first that this set is in the class $\Pi^1_2$. It suffices, as in the case of Turing machines, to write that $L(\mathcal{C}_z) = \Omega^\omega$ if and only if "$\forall\, \sigma \in \Omega^\omega\ \exists r \in \{0, 1\}^\omega$ ($r$ is an accepting run of $\mathcal{C}_z$ over $\sigma$)". The two quantifiers of type 1 are followed by an arithmetical formula. Thus $\{z \in \mathbb{N} \mid L(\mathcal{C}_z) = \Omega^\omega\}$ is in the class $\Pi^1_2$.
In order to prove completeness we shall use the corresponding result for Turing machines proved in [CC89]: the set $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\}$ is $\Pi^1_2$-complete. Consider now the injective computable function $H$ from $\mathbb{N}$ into $\mathbb{N}$ defined above. We are going to prove that, for each integer $z \in \mathbb{N}$, it holds that

$$L(\mathcal{M}_z) = \Sigma^\omega \text{ if and only if } L(\mathcal{C}_{H(z)}) = \Omega^\omega.$$

By Lemma 3.7, for each integer $z \in \mathbb{N}$, if $A_{H_2 \circ H_1(z)}$ is the real time Büchi 8-counter automaton of index $H_2 \circ H_1(z)$, then : $L(A_{H_2 \circ H_1(z)}) = \theta_S(L(\mathcal{M}_z)) \cup \theta_S(\Sigma^\omega)^-$. Thus $L(\mathcal{M}_z) = \Sigma^\omega$ iff $L(A_{H_2 \circ H_1(z)}) = (\Sigma \cup \{E\})^\omega$.
Next applying Lemma 3.8 we see that

$$L(\mathcal{C}_{H_3 \circ H_2 \circ H_1(z)}) = \phi_K(h_K(L(A_{H_2 \circ H_1(z)})) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, 0\})^\omega)^-$$

Thus $L(\mathcal{C}_{H_3\circ H_2\circ H_1(z)}) = \Omega^\omega$
$\leftrightarrow \phi_K(h_K(L(A_{H_2\circ H_1(z)})) \cup h_K(\Gamma^\omega)^-) = \phi_K((\Gamma \cup \{A, B, 0\})^\omega)$
$\leftrightarrow h_K(L(A_{H_2\circ H_1(z)})) \cup h_K(\Gamma^\omega)^- = (\Gamma \cup \{A, B, 0\})^\omega$
$\leftrightarrow L(A_{H_2\circ H_1(z)}) = \Gamma^\omega$
$\leftrightarrow L(\mathcal{M}_z) = \Sigma^\omega$.
This shows that $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) = \Omega^\omega\}$. Thus this latter set is $\Pi^1_2$-complete. □

**Remark 3.10.** *An easy coding can be used to show that the above result still holds if we replace the alphabet $\Omega$ by a two letter alphabet (or even by an alphabet containing $n$ letters for an integer $n \geq 2$). This will be true for all the results presented in this paper.*

**Remark 3.11.** *If we consider context-free languages accepted by Büchi pushdown automata, it is easy to see that the universality problem is still in the class $\Pi^1_2$. Then we can infer from Theorem 3.9 the following corollary.*

**Corollary 3.12.** *The universality problem for context-free $\omega$-languages accepted by Büchi pushdown automata is $\Pi^1_2$-complete.*

Using a similar method as in the proof of Theorem 3.9, we can prove the following result:

**Theorem 3.13.** *The cofiniteness problem for $\omega$-languages of real time Büchi 1-counter automata is $\Pi^1_2$-complete, i.e. the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is cofinite $\}$ is $\Pi^1_2$-complete.*

**Proof.** We first prove that the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is cofinite $\}$ is in the class $\Pi^1_2$. We can reason as in the corresponding proof for Turing machines in [CC89]. Consider a recursive bijection $b : (\mathbb{N}^\star)^2 \rightarrow \mathbb{N}^\star$ and its inverse $b^{-1}$. Now we can consider an infinite word over a finite alphabet $\Omega$ as a countably infinite family of infinite words over the same alphabet by considering, for any $\omega$-word $\sigma \in \Omega^\omega$, the family of $\omega$-words $(\sigma_i)$ suh that for each $i \geq 1$, the $\omega$-word $\sigma_i \in \Omega^\omega$ is defined by $\sigma_i(j) = \sigma(b(i, j))$ for each $j \geq 1$.
We can now express that $L(\mathcal{C}_z)$ is cofinite by a formula :
" $\forall\ \sigma \in \Omega^\omega\ \exists r \in \{0, 1\}^\omega\ \exists i \geq 1$ [ if (all $\omega$-words $\sigma_i$, $i \geq 1$, are distinct), then ($r$ is an accepting run of $\mathcal{C}_z$ over $\sigma_i$) ]".
This is a $\Pi^1_2$-formula because "all $\omega$-words $\sigma_i$ are distinct" can be expressed by the arithmetical formula : $(\forall j, k \geq 1)(\exists i \geq 1)\ \sigma_j(i) \neq \sigma_k(i)$.

To prove that the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is cofinite $\}$ is $\Pi^1_2$-complete, it suffices to remark that $L(\mathcal{M}_z)$ is cofinite if and only if $L(\mathcal{C}_{H_3\circ H_2\circ H_1(z)}) =$ is cofinite. Thus

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) \text{ is cofinite }\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is cofinite }\}$$

So the completeness follows from the fact, proved in [CC89], that the set $\{z \in \mathbb{N} \mid L(\mathcal{M}_z)$ is cofinite $\}$ is $\Pi^1_2$-complete. □

As for the universality problem, we obtain the same complexity when considering context-free $\omega$-languages.

**Corollary 3.14.** *The cofiniteness problem for context-free $\omega$-languages accepted by Büchi pushdown automata is $\Pi_2^1$-complete.*

We now determine the exact complexities of the inclusion and the equivalence problems for $\omega$-languages of real time Büchi 1-counter automata.

**Theorem 3.15.** *The inclusion and the equivalence problems for $\omega$-languages of real time Büchi 1-counter automata are also $\Pi_2^1$-complete, i.e. :*

(1) $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)\}$ *is $\Pi_2^1$-complete.*
(2) $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) = L(\mathcal{C}_z)\}$ *is $\Pi_2^1$-complete.*

**Proof.** We first prove that the set $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)\}$ is a $\Pi_2^1$-set. It suffices to remark that "$L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)$" can be expressed by the $\Pi_2^1$-formula : " $\forall\ \sigma \in \Omega^\omega\ \forall r \in \{0,1\}^\omega\ \exists r' \in \{0,1\}^\omega$ [ if ($r$ is an accepting run of $\mathcal{C}_y$ over $\sigma$), then ($r'$ is an accepting run of $\mathcal{C}_z$ over $\sigma$) ]".
Then the set $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) = L(\mathcal{C}_z)\}$ which is the intersection of the two sets $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)\}$ and $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_z) \subseteq L(\mathcal{C}_y)\}$ is also a $\Pi_2^1$-set.
To prove completeness we denote $n_0$ the index of a real time Büchi 1-counter automaton accepting the $\omega$-language $\Omega^\omega$. Then we consider the function $F : \mathbb{N} \to \mathbb{N}^2$ defined by $F(z) = (n_0, z)$. This function is injective and computable and for all $z \in \mathbb{N}$ it holds that $L(\mathcal{C}_z) = \Omega^\omega$ iff $F(z) = (n_0, z) \in \{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)\}$. Thus Theorem 3.9 implies that $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)\}$ is $\Pi_2^1$-complete.
In a similar way, we prove that the set $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{C}_y) = L(\mathcal{C}_z)\}$ is $\Pi_2^1$-complete. □

As for the previous results we easily get the following corollaries.

**Corollary 3.16.** *The inclusion and the equivalence problems for context-free $\omega$-languages accepted by Büchi pushdown automata are $\Pi_2^1$-complete.*

A natural question about 1-counter $\omega$-languages or context-free $\omega$-languages is the following one : "can we decide whether a given 1-counter $\omega$-language (respectively, context-free $\omega$-language) is regular, i.e. accepted by a Büchi automaton ?". We can state the following result.

**Theorem 3.17.** *The regularity problem for $\omega$-languages of real time Büchi 1-counter automata is $\Pi_2^1$-complete, i.e. : the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is regular $\}$ is $\Pi_2^1$-complete.*

**Proof.** We first prove that the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is regular $\}$ is in the class $\Pi_2^1$. We denote $R_C$ the set of indices of real time Büchi 1-counter automata such that no transition of these automata change the counter value. So the counter value of these automata is always zero and they can be seen simply as Büchi automata. The set $R_C$ is obviously recursive and we can express "$L(\mathcal{C}_z)$ is regular " by the formula : $\exists y[\ y \in R_C$ and $L(\mathcal{C}_z) = L(\mathcal{C}_y)$ ]. The existential quantification is of type

0 and we have already seen that $L(\mathcal{C}_z) = L(\mathcal{C}_y)$ can be expressed by a $\Pi_2^1$-formula. This proves that the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is regular $\}$ is in the class $\Pi_2^1$.

In order to prove the completeness, we shall use the following result of [CC89]. The set $P_{recursive} = \{z \in \mathbb{N} \mid \exists y\ L(\mathcal{M}_z)^- = L(\mathcal{M}_y)\}$ is $\Pi_2^1$-complete.

In fact Castro and Cucker defined a injective computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that :
(1) if $z \in U_2^-$ then $L(\mathcal{M}_{\varphi(z)}) = \Sigma^\omega$ (and so $\varphi(z) \in P_{recursive}$), and
(2) if $z \in U_2$ then $\varphi(z) \notin P_{recursive}$.

Similarly we shall consider the function $H \circ \varphi$ which is an injective and computable function from $\mathbb{N}$ into $\mathbb{N}$. And we are going to show that :
(1) if $z \in U_2^-$ then $L(\mathcal{C}_{H\circ\varphi(z)}) = \Omega^\omega$, and
(2) if $z \in U_2$ then $L(\mathcal{C}_{H\circ\varphi(z)})$ is not a regular $\omega$-language.

We consider now two cases.
**First case.** If $z \in U_2^-$ then $L(\mathcal{M}_{\varphi(z)}) = \Sigma^\omega$ so $L(\mathcal{C}_{H\circ\varphi(z)}) = \Omega^\omega$. Thus in this case $L(\mathcal{C}_{H\circ\varphi(z)})$ is a regular $\omega$-language.
**Second case.** If $z \in U_2$ then $\varphi(z) \notin P_{recursive}$, i.e. $L(\mathcal{M}_{\varphi(z)})^-$ is not accepted by any Turing machine with $1'$ (or Büchi) acceptance condition. It is then easy to see that $L(\mathcal{C}_{H\circ\varphi(z)})^-$ is not accepted by any Turing machine with $1'$ (or Büchi) acceptance condition. Indeed if we denote again $A_{H_2\circ H_1\circ\varphi(z)}$ the real time Büchi 8-counter automaton of index $H_2 \circ H_1 \circ \varphi(z)$, then : $L(A_{H_2\circ H_1\circ\varphi(z)}) = \theta_S(L(\mathcal{M}_{\varphi(z)})) \cup \theta_S(\Sigma^\omega)^-$. Thus $L(A_{H_2\circ H_1\circ\varphi(z)})^- = \theta_S(\Sigma^\omega) - \theta_S(L(\mathcal{M}_{\varphi(z)})) = \theta_S(L(\mathcal{M}_{\varphi(z)})^-)$ is not accepted by any Turing machine with $1'$ (or Büchi) acceptance condition. Next we see that

$$L(\mathcal{C}_{H\circ\varphi(z)}) = \phi_K(h_K(L(A_{H_2\circ H_1\circ\varphi(z)})) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, 0\})^\omega)^-$$

so its complement

$$L(\mathcal{C}_{H\circ\varphi(z)})^- = \phi_K(h_K(L(A_{H_2\circ H_1\circ\varphi(z)})^-))$$

is not accepted by any Turing machine with $1'$ (or Büchi) acceptance condition. In particular $L(\mathcal{C}_{H\circ\varphi(z)})$ is not a regular $\omega$-language because otherwise its complement would be also regular hence accepted by a Turing machine.

Finally, using the reduction $H \circ \varphi$, we have proved that :

$$U_2^- \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is regular }\}$$

and this proves that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is regular $\}$ is $\Pi_2^1$-complete. □

We have also the following result about context-free $\omega$-languages.

**Corollary 3.18.** *The regularity problem for context-free $\omega$-languages accepted by Büchi pushdown automata is $\Pi_2^1$-complete.*

We consider now the complementability problem and the determinizability problems. The complementability problem is $\Pi_2^1$-complete for $\omega$-languages of Turing machines, i. e. the set $P_{recursive} = \{z \in \mathbb{N} \mid \exists y\ L(\mathcal{M}_z)^- = L(\mathcal{M}_y)\}$ is $\Pi_2^1$-complete, [CC89]. We are going to show that it is also $\Pi_2^1$-complete for $\omega$-languages of real time Büchi 1-counter automata or of Büchi pushdown automata. We show also that the determinizability problems for $\omega$-languages of real time Büchi 1-counter automata, or of Büchi pushdown automata, are $\Pi_2^1$-complete. We denote $D_C$ the set of indices of *deterministic* real time Büchi 1-counter automata. We can now state the following result:

**Theorem 3.19.** *The complementability problem and the determinizability problem for $\omega$-languages of real time Büchi 1-counter automata are $\Pi_2^1$-complete, i.e. :*

(1) $\{z \in \mathbb{N} \mid \exists y\ L(\mathcal{C}_z)^- = L(\mathcal{C}_y)\}$ *is $\Pi_2^1$-complete.*
(2) $\{z \in \mathbb{N} \mid \exists y \in D_C\ L(\mathcal{C}_z) = L(\mathcal{C}_y)\}$ *is $\Pi_2^1$-complete.*

**Proof.** We first show that all these problems are in the class $\Pi_2^1$. It is easy to see that $\{z \in \mathbb{N} \mid \exists y L(\mathcal{C}_z)^- = L(\mathcal{C}_y)\}$ is in the class $\Pi_2^1$ because $L(\mathcal{C}_z)^- = L(\mathcal{C}_y)$ can be expressed by a $\Pi_2^1$-formula and the quantification $\exists y$ is of type 0.
On the other hand, it is easy to see that the set $D_C$ is recursive. The formula $\exists y \in D_C\ L(\mathcal{C}_z) = L(\mathcal{C}_y)$ can be written : "$\exists y[y \in D_C$ and $L(\mathcal{C}_z) = L(\mathcal{C}_y)]$" and it can be expressed by a $\Pi_2^1$-formula because the quantification $\exists y$ is of type 0 and $L(\mathcal{C}_z) = L(\mathcal{C}_y)$ can be expressed by a $\Pi_2^1$-formula. Thus the set $\{z \in \mathbb{N} \mid \exists y \in D_C\ L(\mathcal{C}_z) = L(\mathcal{C}_y)\}$ is in the class $\Pi_2^1$.

Consider now the reduction $H \circ \varphi$ already considered in the proof of Theorem 3.17. We have seen that there are two cases.
**First case.** If $z \in U_2^-$ then $L(\mathcal{M}_{\varphi(z)}) = \Sigma^\omega$ so $L(\mathcal{C}_{H\circ\varphi(z)}) = \Omega^\omega$. In this case $L(\mathcal{C}_{H\circ\varphi(z)})$ is obviously accepted by a deterministic real time Büchi 1-counter automaton. Moreover its complement is empty therefore it is also accepted by a real time Büchi 1-counter automaton.
**Second case.** If $z \in U_2$ then $\varphi(z) \notin P_{recursive}$, and $L(\mathcal{C}_{H\circ\varphi(z)})^-$ is not accepted by any Turing machine with $1'$ (or Büchi) acceptance condition. In particular, $L(\mathcal{C}_{H\circ\varphi(z)})^-$ is not accepted by any real time Büchi 1-counter automaton. And $L(\mathcal{C}_{H\circ\varphi(z)})$ can not be accepted by any deterministic real time Büchi 1-counter automaton because otherwise it would be in the arithmetical class $\Pi_2$ (see [Sta97]) and its complement would be accepted by a Turing machine with $1'$ (or Büchi) acceptance condition.

This proves that :

$$U_2^- \leq_1 \{z \in \mathbb{N} \mid \exists y\ L(\mathcal{C}_z)^- = L(\mathcal{C}_y)\}$$

and

$$U_2^- \leq_1 \{z \in \mathbb{N} \mid \exists y \in D_C\ L(\mathcal{C}_z) = L(\mathcal{C}_y)\}$$

and this ends the proof. □

In a similar manner we prove the following result about context-free $\omega$-languages.

**Corollary 3.20.** *The complementability problem and the determinizability problem for context-free $\omega$-languages accepted by Büchi pushdown automata are $\Pi_2^1$-complete.*

We investigate now the unambiguity problem for $\omega$-languages accepted by real time Büchi 1-counter automata or by Büchi pushdown automata. Recall that a real time Büchi 1-counter automaton $\mathcal{A}$, accepting infinite words over an alphabet $\Omega$, is said to be non ambiguous iff for every $\omega$-word $x \in \Omega^\omega$ there is at most one accepting run of $\mathcal{A}$ on $x$. An $\omega$-language $L(\mathcal{A})$, accepted by a real time Büchi 1-counter automaton $\mathcal{A}$, is said to be non ambiguous iff there exists a non ambiguous real time Büchi 1-counter automaton $\mathcal{B}$ such that $L(\mathcal{B}) = L(\mathcal{A})$; in the other case the $\omega$-language $L(\mathcal{A})$ is said to be inherently ambiguous (notice that the notion of ambiguity refer here to acceptance by real time Büchi 1-counter automata). The definition is similar for $\omega$-languages accepted by Büchi pushdown automata. A context-free $\omega$-language $L$ is said to be non ambiguous iff there exists a non ambiguous Büchi pushdown automaton accepting $L$. It has been proved in [Fin03a] that one cannot decide whether a given context-free $\omega$-language $L$ is non ambiguous. We now state the following result.

**Theorem 3.21.** *The unambiguity problem for $\omega$-languages of real time Büchi 1-counter automata is $\Pi_2^1$-complete, i.e. :*

$$\textit{The set } \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \textit{ is non ambiguous } \} \textit{ is } \Pi_2^1\textit{-complete.}$$

**Proof.** We can first express "$\mathcal{C}_z$ is non ambiguous" by :

$$\text{“}\forall \sigma \in \Omega^\omega \forall r, r' \in \{0,1\}^\omega [(r \text{ and } r' \text{ are accepting runs of } \mathcal{C}_z \text{ on } \sigma) \rightarrow r = r']\text{”}$$

which is a $\Pi_1^1$-formula. Then " $L(\mathcal{C}_z)$ is non ambiguous" can be expressed by the following formula: "$\exists y[L(\mathcal{C}_z) = L(\mathcal{C}_y)$ and $\mathcal{C}_y$ is non ambiguous]". This is a $\Pi_2^1$-formula because $L(\mathcal{C}_z) = L(\mathcal{C}_y)$ can be expressed by a $\Pi_2^1$-formula, and the quantification $\exists y$ is of type 0. Thus the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is non ambiguous $\}$ is a $\Pi_2^1$-set.

To prove completeness we shall use the following result proved in [FS03]. Let $L(\mathcal{A})$ be a context-free $\omega$-language accepted by a Büchi pushdown automaton $\mathcal{A}$ such that $L(\mathcal{A})$ is an analytic but non Borel set. Then the set of $\omega$-words, which have $2^{\aleph_0}$ accepting runs by $\mathcal{A}$, has cardinality $2^{\aleph_0}$. In particular $L(\mathcal{A})$ has the maximum degree of ambiguity; it is said to be inherently ambiguous of degree $2^{\aleph_0}$ in [Fin03a].

We define the following simple operations over $\omega$-languages. For two $\omega$-words $x, x' \in \Sigma^\omega$ the $\omega$-word $x \otimes x'$ is defined by : for every integer $n \geq 1$ $(x \otimes x')(2n-1) = x(n)$ and $(x \otimes x')(2n) = x'(n)$. For two $\omega$-languages $L, L' \subseteq \Sigma^\omega$, the $\omega$-language $L \otimes L'$ is defined by $L \otimes L' = \{x \otimes x' \mid x \in L$ and $x' \in L'\}$.

We shall in the sequel use the following construction. We know that there is a simple example of $\mathbf{\Sigma}_1^1$-complete set $L \subseteq \Sigma^\omega$ accepted by a 1-counter automaton, hence by a Turing machine with $1'$ acceptance condition, see [Fin03b]. Then it is easy to define an injective computable function $\theta$ from $\mathbb{N}$ into $\mathbb{N}$ such that, for every integer $z \in \mathbb{N}$, it holds that $L(\mathcal{M}_{\theta(z)}) = (L \otimes \Sigma^\omega) \cup (\Sigma^\omega \otimes L(\mathcal{M}_z))$.

We are going to use now the reduction $H$ already considered above to show that the universality problem for $\omega$-languages of real time Büchi 1-counter automata is $\Pi_2^1$-complete. We have seen that

$$L(\mathcal{M}_z) = \Sigma^\omega \text{ if and only if } L(\mathcal{C}_{H(z)}) = \Omega^\omega$$

and we can easily see that

$$L(\mathcal{M}_{\theta(z)}) = \Sigma^\omega \text{ if and only if } L(\mathcal{M}_z) = \Sigma^\omega$$

because $L \neq \Sigma^\omega$.

The reduction $H \circ \theta$ is an injective computable function from $\mathbb{N}$ into $\mathbb{N}$.
We consider now two cases.
**First case.** $L(\mathcal{M}_z) = \Sigma^\omega$. Then $L(\mathcal{M}_{\theta(z)}) = \Sigma^\omega$ and $L(\mathcal{C}_{H\circ\theta(z)}) = \Omega^\omega$. In particular $L(\mathcal{C}_{H\circ\theta(z)})$ is accepted by a non ambiguous real time Büchi 1-counter automaton.
**Second case.** $L(\mathcal{M}_z) \neq \Sigma^\omega$. Then there is an $\omega$-word $x \in \Sigma^\omega$ such that $x \notin L(\mathcal{M}_z)$. But $L(\mathcal{M}_{\theta(z)}) = (L \otimes \Sigma^\omega) \cup (\Sigma^\omega \otimes L(\mathcal{M}_z))$ thus $\{\sigma \in \Sigma^\omega \mid \sigma \otimes x \in L(\mathcal{M}_{\theta(z)})\} = L$ is a $\mathbf{\Sigma}_1^1$-complete set. This implies that $L(\mathcal{M}_{\theta(z)})$ is not a Borel set because otherwise its section $\{\sigma \in \Sigma^\omega \mid \sigma \otimes x \in L(\mathcal{M}_{\theta(z)})\}$ would be also Borel, [Kec95].
Recall that $H = H_3 \circ H_2 \circ H_1$, where $H_1$, $H_2$, and $H_3$ are the computable functions from $\mathbb{N}$ into $\mathbb{N}$ defined above. If $A_{H_2\circ H_1\circ\theta(z)}$ is the real time Büchi 8-counter automaton of index $H_2 \circ H_1 \circ \theta(z)$, then it is easy to see that $L(A_{H_2\circ H_1\circ\theta(z)}) = \theta_S(L(\mathcal{M}_{\theta(z)})) \cup \theta_S(\Sigma^\omega)^-$ is not Borel. Next, considering the mappings $h_K$ and $\phi_K$, we can easily successively see that
$h_K(L(A_{H_2\circ H_1\circ\theta(z)})) \cup h_K(\Gamma^\omega)^-$ is not a Borel set,
$\phi_K(h_K(L(A_{H_2\circ H_1\circ\theta(z)})) \cup h_K(\Gamma^\omega)^-)$ is not a Borel set,
$L(\mathcal{C}_{H_3\circ H_2\circ H_1\circ\theta(z)}) = \phi_K(h_K(L(A_{H_2\circ H_1\circ\theta(z)})) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, 0\})^\omega)^-$ is not a Borel set, i.e. the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is not a Borel set.
Thus in that case the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is inherently ambiguous (and it is even inherently ambiguous of degree $2^{\aleph_0}$) , [Fin03a].

Finally, using the reduction $H \circ \theta$, we have proved that :

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is non ambiguous }\}$$

Thus this latter set is $\Pi_2^1$-complete. □

In a similar manner we prove the following result about context-free $\omega$-languages.

**Corollary 3.22.** *The unambiguity problem for context-free $\omega$-languages accepted by Büchi pushdown automata is $\Pi^1_2$-complete.*

A fundamental result due to Landweber is that one can determine in an effective manner the topological complexity of regular $\omega$-languages: one can decide whether a given regular $\omega$-language is in a given Borel class (recall that all regular $\omega$-languages belong to the class $\mathbf{\Delta}^0_3$), [Lan69]. The question naturally arises of a similar problem for other classes of languages, like $\omega$-languages of real time Büchi 1-counter automata. It is proved in [Fin06a] that $\omega$-languages of real time Büchi 1-counter automata have the same topological complexity as $\omega$-languages of Turing machines. From the above proof we can now infer that the topological complexity of $\omega$-languages of real time Büchi 1-counter automata is highly undecidable.

**Theorem 3.23.** *Let $\alpha$ be a countable ordinal. Then*
1. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the Borel class* $\mathbf{\Sigma}^0_\alpha\}$ *is* $\Pi^1_2$*-hard.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the Borel class* $\mathbf{\Pi}^0_\alpha\}$ *is* $\Pi^1_2$*-hard.*
3. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is a Borel set* $\}$ *is* $\Pi^1_2$*-hard.*

**Proof.** We can use the same reduction $H \circ \theta$ as in the proof of Theorem 3.21. We have seen that there are two cases.
**First case.** $L(\mathcal{M}_z) = \Sigma^\omega$. Then $L(\mathcal{M}_{\theta(z)}) = \Sigma^\omega$ and $L(\mathcal{C}_{H\circ\theta(z)}) = \Omega^\omega$. In particular $L(\mathcal{C}_{H\circ\theta(z)})$ is an open and closed subset of $\Omega^\omega$ and it belongs to all Borel classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.
**Second case.** $L(\mathcal{M}_z) \neq \Sigma^\omega$. Then we have seen that the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is not a Borel set.

Finally, using the reduction $H \circ \theta$, we have proved that :

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the Borel class } \mathbf{\Sigma}^0_\alpha\}$$

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the Borel class } \mathbf{\Pi}^0_\alpha\}$$
$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is a Borel set }\}$$
And this ends the proof since $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\}$ is $\Pi^1_2$-complete. □

In the case of context-free $\omega$-languages accepted by Büchi pushdown automata the corresponding problems have been shown to be undecidable, using the undecidability of the Post correspondence problem [Fin01, Fin03b]. We can prove as above that they are in fact highly undecidable.

**Corollary 3.24.** *Let $\alpha$ be a countable ordinal. The following problems are $\Pi^1_2$-hard.*
1. *"Determine whether a given context-free $\omega$-language is in the Borel class $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$)".*
2. *"Determine whether a given context-free $\omega$-language is a Borel set".*

**Remark 3.25.** *If $\alpha$ is an ordinal smaller than the Church-Kleene ordinal $\omega_1^{\mathrm{CK}}$, i.e. is a recursive ordinal, then there exists a universal set for $\mathbf{\Sigma}^0_\alpha$-subsets of $X^\omega$ which is in the class $\Delta^1_1$. This is a known fact of Effective Descriptive Set Theory which is proved in detail in [FL07]. This means that there exists a $\Delta^1_1$-set $U_\alpha \subseteq 2^\omega \times X^\omega$ such that for every set $L \subseteq X^\omega$, $L$ is in the class $\mathbf{\Sigma}^0_\alpha$ iff there is an $\omega$-word $x \in 2^\omega$ such that $[\forall y \in X^\omega \ y \in L \leftrightarrow (x, y) \in U_\alpha]$, i.e. such that $L$ is the section of $U_\alpha$ in $x$. The $\Delta^1_1$-set $U_\alpha \subseteq 2^\omega \times X^\omega$ is accepted by a Turing machine with $1'$ or Büchi acceptance condition. Then we can prove that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\mathbf{\Sigma}^0_\alpha\}$ is in fact a $\Sigma^1_3$-set. Similarly the existence of a $\Delta^1_1$ universal set for $\mathbf{\Pi}^0_\alpha$-subsets of $X^\omega$ implies that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\mathbf{\Pi}^0_\alpha\}$ is in fact a $\Sigma^1_3$-set. Similar results hold for context-free $\omega$-languages accepted by Büchi pushdown automata.*

We consider now the arithmetical complexity of $\omega$-languages of real time Büchi 1-counter automata. Here we get the exact complexity of highly undecidable problems.

**Theorem 3.26.** *Let $n \geq 1$ be an integer. Then*

1. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the arithmetical class* $\Sigma_n\}$ *is* $\Pi^1_2$*-complete.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is in the arithmetical class* $\Pi_n\}$ *is* $\Pi^1_2$*-complete.*
3. $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ *is a* $\Delta^1_1$*-set* $\}$ *is* $\Pi^1_2$*-complete.*

**Proof.** Let $n \geq 1$ be an integer. We first prove that

$$\{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the arithmetical class } \Sigma_n\}$$

is a $\Pi^1_2$-set. We are going to use the existence of a universal set $\mathcal{U}_{\Sigma_n} \subseteq \mathbb{N} \times \Omega^\omega$ for the class of $\Sigma_n$-subsets of $\Omega^\omega$, [Mos80, p. 172]. The set $\mathcal{U}_{\Sigma_n}$ is a $\Sigma_n$-subset of $\mathbb{N} \times \Omega^\omega$ (i.e. $(n, x) \in \mathcal{U}_{\Sigma_n}$ can be expressed by a $\Sigma^0_n$-formula) and for any $L \subseteq \Omega^\omega$, $L$ is a $\Sigma_n$-set iff there is an integer $n$ such that $[\forall x \in \Omega^\omega \ x \in L \leftrightarrow (n, x) \in \mathcal{U}_{\Sigma_n}]$.
Then we can express "$L(\mathcal{C}_z)$ is in the arithmetical class $\Sigma_n$" by the formula "$\exists n \in \mathbb{N} \ \forall x \in \Omega^\omega \ [x \in L(\mathcal{C}_z) \leftrightarrow (n, x) \in \mathcal{U}_{\Sigma_n}]$". The formula "$[x \in L(\mathcal{C}_z) \leftrightarrow (n, x) \in \mathcal{U}_{\Sigma_n}]$" is a $\Delta^1_2$-formula and the first quantifier $\exists$ is of type 0. Therefore "$L(\mathcal{C}_z)$ is in the arithmetical class $\Sigma_n$" can be expressed by a $\Pi^1_2$-formula.
The case of the arithmetical class $\Pi_n$ is very similar since there exists also a universal set $\mathcal{U}_{\Pi_n} \subseteq \mathbb{N} \times \Omega^\omega$ for the class of $\Pi_n$-subsets of $\Omega^\omega$, [Mos80].
We now prove that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is a $\Delta^1_1$-set $\}$ is a $\Pi^1_2$-set. We have already seen that the set $P_{recursive} = \{z \in \mathbb{N} \mid \exists y \ L(\mathcal{M}_z)^- = L(\mathcal{M}_y)\}$ is $\Pi^1_2$-complete, [CC89]. On the other hand, an $\omega$-language $L \subseteq X^\omega$ is in the class $\Sigma^1_1$ iff it is accepted by a non deterministic Turing machine with a $1'$ or Büchi acceptance condition, [Sta97]. Thus $P_{recursive} = \{z \in \mathbb{N} \mid L(\mathcal{M}_z)$ is a $\Delta^1_1$-set $\}$. In a similar manner, $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is a $\Delta^1_1$-set $\} = \{z \in \mathbb{N} \mid \exists y \ L(\mathcal{C}_z)^- = L(\mathcal{M}_y)\}$, and it is easily seen to be in the class $\Pi^1_2$.

We now prove completeness for the three problems. We can again use the same reduction $H \circ \theta$ as in the proof of Theorem 3.21. We have seen that there are two cases.

**First case.** $L(\mathcal{M}_z) = \Sigma^\omega$. Then $L(\mathcal{M}_{\theta(z)}) = \Sigma^\omega$ and $L(\mathcal{C}_{H\circ\theta(z)}) = \Omega^\omega$. In particular, for every integer $n \geq 1$, the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is in the arithmetical classes $\Sigma_n$ and $\Pi_n$.
**Second case.** $L(\mathcal{M}_z) \neq \Sigma^\omega$. Then we have seen that the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is not a Borel set. Thus it is not a (lightface) $\Delta_1^1$-set and it is not in any arithmetical class $\Sigma_n$ or $\Pi_n$.

Finally, using the reduction $H \circ \theta$, we have proved that :

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the arithmetical class } \Sigma_n\}$$

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is in the arithmetical class } \Pi_n\}$$
$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{C}_z) \text{ is a } \Delta_1^1\text{-set }\}$$

And this ends the proof since $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\}$ is $\Pi_2^1$-complete. □

In a similar way, we can prove the following result for context-free $\omega$-languages accepted by Büchi pushdown automata. Notice that the decision problems cited in the following corollary were shown to be undecidable in [Fin01, Fin03b] but their exact (high) complexity was unexpected.

**Corollary 3.27.** *Let $n \geq 1$ be an integer. The following decision problems are $\Pi_2^1$-complete.*

1. *"Determine whether a given context-free $\omega$-language is in the arithmetical class $\Sigma_n$ (respectively, $\Pi_n$)"*
2. *"Determine whether a given context-free $\omega$-language is a $\Delta_1^1$-set".*

## 4. Infinite computations of 2-tape automata

We are going to study now decision problems about the infinite behaviour of 2-tape Büchi automata accepting infinitary rational relations. We first recall the definition of 2-tape Büchi automata and of infinitary rational relations.

**Definition 4.1.** *A 2-tape Büchi automaton is a sextuple $\mathcal{T} = (K, \Sigma_1, \Sigma_2, \Delta, q_0, F)$, where $K$ is a finite set of states, $\Sigma_1$ and $\Sigma_2$ are finite alphabets, $\Delta$ is a finite subset of $K \times \Sigma_1^\star \times \Sigma_2^\star \times K$ called the set of transitions, $q_0$ is the initial state, and $F \subseteq K$ is the set of accepting states.*
*A computation $\mathcal{C}$ of the 2-tape Büchi automaton $\mathcal{T}$ is an infinite sequence of transitions*

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

*The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$.*
*The input word of the computation is $u = u_1.u_2.u_3\ldots$*
*The output word of the computation is $v = v_1.v_2.v_3\ldots$*

*Then the input and the output words may be finite or infinite.*
*The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ accepted by the 2-tape Büchi automaton $\mathcal{T}$ is the set of couples $(u, v) \in \Sigma_1^\omega \times \Sigma_2^\omega$ such that $u$ and $v$ are the input and the output words of some successful computation $\mathcal{C}$ of $\mathcal{T}$.*
*The set of infinitary rational relations will be denoted $\mathbf{RAT}_\omega$.*

In order to prove that some decision problems about the infinite behaviour of 2-tape Büchi automata are highly undecidable, we shall use the results of the preceding section and a coding used in a previous paper on the topological complexity of infinitary rational relations. We proved in [Fin06b] that infinitary rational relations have the same topological complexity as $\omega$-languages accepted by Büchi Turing machines. This very surprising result was obtained by using a simulation of the behaviour of real time 1-counter automata by 2-tape Büchi automata. We recall now a coding which was used in [Fin06b].

We now first define a coding of an $\omega$-word over the finite alphabet $\Omega = \{a, b, E, A, B, F, 0\}$ by an $\omega$-word over the alphabet $\Omega' = \Omega \cup \{C\}$, where $C$ is an additionnal letter not in $\Omega$.

For $x \in \Omega^\omega$ the $\omega$-word $h(x)$ is defined by :

$$h(x) = C.0.x(1).C.0^2.x(2).C.0^3.x(3).C \ldots C.0^n.x(n).C.0^{n+1}.x(n+1).C \ldots$$

Then it is easy to see that the mapping $h$ from $\Omega^\omega$ into $(\Omega \cup \{C\})^\omega$ is continuous and injective.

Let now $\alpha$ be the $\omega$-word over the alphabet $\Omega'$ which is simply defined by:

$$\alpha = C.0.C.0^2.C.0^3.C.0^4.C \ldots C.0^n.C.0^{n+1}.C \ldots$$

The following results were proved in [Fin06b].

**Lemma 4.2.** *Let $\Omega$ be a finite alphabet such that $0 \in \Omega$, $\alpha$ be the $\omega$-word over $\Omega \cup \{C\}$ defined as above, and $L \subseteq \Omega^\omega$ be in $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$. Then there exists an infinitary rational relation $R_1 \subseteq (\Omega \cup \{C\})^\omega \times (\Omega \cup \{C\})^\omega$ such that:*

$$\forall x \in \Omega^\omega \ (x \in L) \textit{ iff } ((h(x), \alpha) \in R_1)$$

**Lemma 4.3.** *The set $R_2 = (\Omega \cup \{C\})^\omega \times (\Omega \cup \{C\})^\omega - (h(\Omega^\omega) \times \{\alpha\})$ is an infinitary rational relation.*

Considering the union $R_1 \cup R_2$ of the two infinitary rational relations obtained in the two above lemmas we get the following result.

**Proposition 4.4.** *Let $L \subseteq \Omega^\omega$ be in $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$ and $\mathcal{L} = h(L) \cup (h(\Omega^\omega))^-$. Then*

$$R = \mathcal{L} \times \{\alpha\} \bigcup (\Omega')^\omega \times ((\Omega')^\omega - \{\alpha\})$$

*is an infinitary rational relation.*

Moreover it is proved in [Fin06b] that one can construct effectively, from a real time 1-counter Büchi automaton $\mathcal{A}$ accepting $L$, a 2-tape Büchi automaton $\mathcal{B}$ accepting the infinitary relation $R = \mathcal{L} \times \{\alpha\} \bigcup (\Omega')^\omega \times ((\Omega')^\omega - \{\alpha\})$.
This can be done in an injective way, so we get the following result.

Notice that from now on we shall denote $\mathcal{T}_z$ the 2-tape Büchi automaton of index $z$.

**Lemma 4.5.** *There is an injective computable function $H'$ from $\mathbb{N}$ into $\mathbb{N}$ satisfying the following property.*
*If $\mathcal{C}_z$ is the real time Büchi 1-counter automaton (reading words over $\Omega$) of index $z$, and if $\mathcal{T}_{H'(z)}$ is the 2-tape Büchi automaton of index $H'(z)$, then : $R(\mathcal{T}_{H'(z)}) = (h(L(\mathcal{C}_z)) \cup (h(\Omega^\omega))^-) \times \{\alpha\} \bigcup (\Omega')^\omega \times ((\Omega')^\omega - \{\alpha\})$.*

We can now state our first results about 2-tape Büchi automata. Notice that the four decision problems considered here were known to be undecidable. But the proof used the undecidability of Post correspondence problem, as in the case of finitary rational relations stated in [Ber79], in such a way that these decision problems were only proved to be hard for the first level of the *arithmetical* hierarchy. We obtain here the exact complexity of these problems which is surprisingly high.

**Theorem 4.6.** *The universality problem, the cofiniteness problem, the equivalence problem, and the inclusion problem for infinitary rational relations are $\Pi^1_2$-complete, i.e. :*

1. $\{z \in \mathbb{N} \mid R(\mathcal{T}_z) = \Omega'^\omega \times \Omega'^\omega\}$ *is $\Pi^1_2$-complete.*
2. $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is cofinite* $\}$ *is $\Pi^1_2$-complete.*
3. $\{(y, z) \in \mathbb{N}^2 \mid R(\mathcal{T}_y) \subseteq R(\mathcal{T}_z)\}$ *is $\Pi^1_2$-complete.*
4. $\{(y, z) \in \mathbb{N}^2 \mid R(\mathcal{T}_y) = R(\mathcal{T}_z)\}$ *is $\Pi^1_2$-complete.*

**Proof.** In order to prove that these problems are in the class $\Pi^1_2$, we can reason as in the case of $\omega$-languages of real time Büchi 1-counter automata.

To prove completeness, we use the reduction $H'$ defined above and the following properties which can be easily checked. For each integer $z$,

1. $L(\mathcal{C}_z) = \Omega^\omega$ iff $R(\mathcal{T}_{H'(z)}) = \Omega'^\omega \times \Omega'^\omega$.
2. $L(\mathcal{C}_z)$ is cofinite iff $R(\mathcal{T}_{H'(z)})$ is cofinite.
3. $L(\mathcal{C}_y) \subseteq L(\mathcal{C}_z)$ iff $R(\mathcal{T}_{H'(y)}) \subseteq R(\mathcal{T}_{H'(z)})$.
4. $L(\mathcal{C}_y) = L(\mathcal{C}_z)$ iff $R(\mathcal{T}_{H'(y)}) = R(\mathcal{T}_{H'(z)})$.

Then the completeness results follow easily from the corresponding results about $\omega$-languages of real time Büchi 1-counter automata, proved in the preceding section. □

We consider now the "regularity problem" for infinitary rational relation. An infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ may be seen as an $\omega$-language over the product alphabet $\Sigma_1 \times \Sigma_2$. Then a relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ is accepted by a Büchi automaton iff it is accepted by a 2-tape Büchi automaton with two reading heads which move synchronously. The relation $R$ is then called a synchronized infinitary

rational relation. These relations have been studied by Frougny and Sakarovitch in [FS93] where they proved that one cannot decide whether a given infinitary rational relation is synchronized. We shall prove that actually this problem is also $\Pi_2^1$-complete. This is also the case for the complementability problem, the determinizability problem, and the unambiguity problem for infinitary rational relations. We denote below $T_D$ the (recursive) set of indices of *deterministic* 2-tape Büchi automata.

**Theorem 4.7.** *The "regularity problem", the complementability problem, the determinizability problem, and the unambiguity problem for infinitary rational relations are $\Pi_2^1$-complete, i.e. :*

(1) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is a synchronized rational relation* $\}$ *is* $\Pi_2^1$*-complete.*
(2) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)^-$ *is an infinitary rational relation* $\}$ *is* $\Pi_2^1$*-complete.*
(3) $\{z \in \mathbb{N} \mid \exists y \in T_D \ R(\mathcal{T}_z) = R(\mathcal{T}_y)\}$ *is* $\Pi_2^1$*-complete.*
(4) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is a non ambiguous rational relation* $\}$ *is* $\Pi_2^1$*-complete.*

**Proof.** We can reason as in the case of $\omega$-languages of real time Büchi 1-counter automata to prove that these problems are in the class $\Pi_2^1$.

To prove completeness we consider the reduction $H \circ \theta$ already used in the proof of Theorem 3.21. And we shall use now the reduction $H' \circ H \circ \theta$, where $H'$ is defined above in this section. The reduction $H' \circ H \circ \theta$ is an injective computable function from $\mathbb{N}$ into $\mathbb{N}$. Returning to the proof of Theorem 3.21, we can see that there are now two cases.

**First case.** $L(\mathcal{M}_z) = \Sigma^\omega$. Then $L(\mathcal{M}_{\theta(z)}) = \Sigma^\omega$ and $L(\mathcal{C}_{H\circ\theta(z)}) = \Omega^\omega$ and $R(\mathcal{T}_{H'\circ H\circ\theta(z)}) = \Omega'^\omega \times \Omega'^\omega$. Thus in that case $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is a synchronized rational relation accepted by a deterministic, hence also non ambiguous, 2-tape Büchi automaton. And its complement is empty so it is also an infinitary rational relation.
**Second case.** $L(\mathcal{M}_z) \neq \Sigma^\omega$. Then we have seen that in that case the $\omega$-language $L(\mathcal{C}_{H\circ\theta(z)})$ is not a Borel set. It is easy to see that the infinitary rational relation $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is also a non Borel set.
Thus in that case $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is not a synchronized rational relation because otherwise it would be a $\mathbf{\Delta}_3^0$-set. The relation $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ can not be accepted by any deterministic 2-tape Büchi automaton because otherwise it would be a $\mathbf{\Pi}_2^0$-set. The relation $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is inherently ambiguous (and it is even inherently ambiguous of degree $2^{\aleph_0}$, see [Fin03a, FS03]). And the complement $\Omega'^\omega \times \Omega'^\omega - R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is not an analytic set (because otherwise $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ would be analytic and coanalytic hence Borel). Thus the complement of $R(\mathcal{T}_{H'\circ H\circ\theta(z)})$ is not an infinitary rational relation.

Finally, using the reduction $H' \circ H \circ \theta$, we have proved that : $\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\}$ is reduced to the four problems we consider here. Thus these problems are $\Pi_2^1$-complete. $\square$

Topological and arithmetical properties of infinitary rational relations have been shown to be undecidable in [Fin03d]. The proofs used the undecidability of Post correspondence problem and the existence of an analytic but non Borel set proved in [Fin03c]. So classical decision problems were only proved to be hard for the first level of the *arithmetical* hierarchy.

We can now infer from the proof of the preceding theorem, reasoning as in the case of $\omega$-languages of real time Büchi 1-counter automata, that topological and arithmetical properties of infinitary rational relations are actually highly undecidable.

**Theorem 4.8.** *Let $\alpha$ be a non null countable ordinal. Then*

(1) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is in the Borel class* $\boldsymbol{\Sigma}^0_\alpha\}$ *is* $\Pi^1_2$*-hard.*
(2) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is in the Borel class* $\boldsymbol{\Pi}^0_\alpha\}$ *is* $\Pi^1_2$*-hard.*
(3) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is a Borel set* $\}$ *is* $\Pi^1_2$*-hard.*

**Theorem 4.9.** *Let $n \geq 1$ be an integer. Then*

(1) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is in the arithmetical class* $\Sigma_n\}$ *is* $\Pi^1_2$*-complete.*
(2) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is in the arithmetical class* $\Pi_n\}$ *is* $\Pi^1_2$*-complete.*
(3) $\{z \in \mathbb{N} \mid R(\mathcal{T}_z)$ *is a* $\Delta^1_1$*-set* $\}$ *is* $\Pi^1_2$*-complete.*

## 5. Concluding remarks and further work

We have got very surprising results which show that many decision problems about $\omega$-languages of real time Büchi 1-counter automata and infinitary rational relations exhibit actually a great complexity, despite the simplicity of the definition of 1-counter automata or 2-tape automata.
Recall that, by Remark 3.25, if $\alpha$ is an ordinal smaller than the Church-Kleene ordinal $\omega_1^{\mathrm{CK}}$, then $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\boldsymbol{\Sigma}^0_\alpha\}$ and $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)$ is in the Borel class $\boldsymbol{\Pi}^0_\alpha\}$ are $\Sigma^1_3$-sets. Moreover they are $\Pi^1_2$-hard by Theorem 3.23. However the exact complexity of being in the Borel class $\boldsymbol{\Sigma}^0_\alpha$ (respectively, $\boldsymbol{\Pi}^0_\alpha$), for a countable ordinal $\alpha$, remains an open problem for $\omega$-languages of real time 1-counter automata (respectively, pushdown automata, 2-tape automata).

May be one of the most surprising results in this paper is that the universality problem for infinitary rational relations accepted by 2-tape Büchi automata is $\Pi^1_2$-complete. This result may be compared to the complexity of the universality problem for timed Büchi automata. Alur and Dill proved in [AD94] that the universality problem for timed Büchi automata is $\Pi^1_1$-hard. On the other hand this problem is known to be in the class $\Pi^1_2$ but its exact complexity is still unknown. Notice that using the $\Pi^1_1$-hardness of the universality problem for timed Büchi automata some other decision problems for timed Büchi automata have been shown to be $\Pi^1_1$-hard, [AD94, Fin06c].

Recognizable languages of infinite bidimensional words (infinite pictures) have been recently studied in [ATW03, Fin04]. Using partly similar reasoning as in this paper we have proved that some decision problems for recognizable languages of infinite pictures have the same degrees as the corresponding problems about $\omega$-languages of real time 1-counter automata, [Fin09]. Notice that some problems, like the non-emptiness problem and the infiniteness problem, are $\Sigma_1^1$-complete for recognizable languages of infinite pictures but are decidable for $\omega$-languages of real time 1-counter automata or 2-tape automata. Some problems studied in [Fin09] are specific to languages of infinite pictures. In particular, it is $\Pi_2^1$-complete to determine whether a given Büchi recognizable language of infinite pictures can be accepted row by row using an automaton model over ordinal words of length $\omega^2$.

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